MPO 663 – Convective and Mesoscale Met.
Homework 2 Solutions
Assigned Feb 21, 2006 – returned Mar 23, 2006

<u>The point of this problem set</u> is to fully understand the *Rossby radius of deformation*, and its dependence on vertical structure. We use the simplest possible framework for a heating-induced balanced vortex: a hydrostatic Boussinesq stratified atmosphere, with perturbations linearized about a state of rest. Thus, $\rho$ = constant (except that there is a buoyancy b), so that the pressure gradient force is written as $\phi_x \equiv p'_x/\rho$. (I used $\pi$ instead of $\phi$ in class, following Houze's notation.)

Before digging into the questions, **understand** our equation set:

| | |
|---|---|
| $u_t = fv - \phi_x$ | u momentum equation |
| $v_t = -fu - \phi_y$ | v momentum equation |
| $0 = b - \phi_z$ | w momentum equation (hydrostatic) |
| $u_x + v_y + w_z = 0$ | continuity equation |
| $b_t = -N^2 w + q$ | thermodynamic equation w/ heating ($q = g/\theta\, d\theta/dt$) |

First, separate vertical and horizontal structure. For this step, it is simplest to go to 2 dimensions (no y variations) and to set aside the Coriolis term, which can be trivially added back in later. This simplifies the equation set to 4. **Understand** them**:**

| | |
|---|---|
| $u_t = -\phi_x$ | u momentum equation |
| $0 = b - \phi_z$ | w momentum equation (hydrostatic) |
| $u_x + w_z = 0$ | continuity equation |
| $b_t = -N^2 w + q$ | thermodynamic equation w/ heating q |

**1.** Assume solutions of the form $u = U(x,t)\cos(mz)$, $\phi = \Phi(x,t)\cos(mz)$, $w = W(x,t)\sin(mz)$, $b = B(x,t)\sin(mz)$, $q = Q(x,t)\sin(mz)$. **Confirm (write** from the confidence of your own knowledge) that these satisfy the boundary condition w=0 at z=0 for all vertical wavenumbers m.

yes, sin(0m) = 0 for all m

**Plug** these forms into the 4 equations above, then **substitute** the hydrostatic equation into the thermo equation and the result into the continuity equation to **show:**

$U_t = -\Phi_x$ u momentum equation
$\Phi_t = -N^2/m^2 (U_x + \delta_d)$ height equation w/ source

where the heating is expressed as *diabatic divergence* $\delta_d \equiv \partial/\partial z(q/N^2) = -mQ/N^2$.

The z derivatives become simple factors of m. The z sinusoids can then be cancelled to give the above.

**Rewrite** these to make clear that they are the equations for forced waves with free-surface height amplitude $h = \Phi/g$, in a shallow (compared to wavelength) layer of water (or any other inviscid fluid), with *equivalent depth* $H = N^2/(gm^2)$.

Substituting h and H as above yields

$U_t = -gh_x$ u momentum equation
$h_t = -H (U_x + \delta_d)$ height equation w/ source

**Confirm** that the v equation, Coriolis terms, and u and v momentum source terms could have been carried along in the same way to give the forced shallow water set for a given value of m:

$U_t = fV - \Phi_x + F$ u momentum equation
$V_t = -fU - \Phi_y + G$ v momentum equation
$\Phi_t = -N^2/m^2 (U_x + V_y + \delta_d)$ how divergence changes $\phi$

Since the u momentum equation passed unchanged through the vertical mode decomposition u = U cos(mz) to give a U equation, a meridional momentum budget v and Coriolis and frictional terms also all pass through without change.

**2.** Now solve for the structure of an inviscid (F=G=0) balanced line vortex (shear line) created by a *past* heating along x = 0 on an f-plane, with $f = 10^{-4}$ $s^{-1}$. Suppose that the heating which created the

vortex was a line of heating along x=0, and only excited a single vertical wavenumber m. **Confirm** that this balanced flow should have no y or t derivatives (i.e., the flow is steady state) with $\delta_d = 0$ at the current time. **Write** the steady state version of the 3-equation set from 1. **What, in words, remains of the equation set's physical meaning**? Does it give us the structure of the vortex? Does it contain the information that we specifically seek the vortex *created by a past heating along x = 0*?

Merely canceling out time derivatives leaves nothing but geostrophic balance. While valid, it is ambiguous, because the height equation's information content is lost (it just becomes 0=0).

**3.** To proceed, we need a different tactic. **Derive** the potential vorticity conservation equation for the 2D shallow water set (with F=G=Q=0), by **constructing** a vorticity equation from U and V and **substituting** divergence from the height equation.

| | |
|---|---|
| $(V_x - U_y)_t = -f(U_x + V_y) \equiv -\zeta_t$ | vorticity equation |
| $h_t = -H\ (U_x + V_y)$ | height equation |
| $\zeta_t\ /f = h_t\ /H$ | PV equation |

**4.** To define our balanced vortex structure, **express mathematically** the statement that PV is constant and equal to its pre-heating value everywhere except along the line x=0 where the heating occurred.

| | |
|---|---|
| $\zeta\ /f = V_x\ /f = h\ /H$ | PV equation, time integrated |
| $V = V_g = g/f\ h_x$ | V is geostrophic |

**Solve** this equation for $\Phi(x)$ and V(x).

| | |
|---|---|
| $V_x = g/f\ h_{xx} = fh/H$ | plug |
| $h_{xx} = f^2/(gH)\ h$ | arrange |
| $h = h_o \exp(-f/\sqrt{(gH)}\ x)$ | solve; pick decaying solution |
| $V = V_g = g/f\ h_x$ | V is geostrophic |

**Write** the horizontal length scale for the vortex's exponential decay structure: this length is called the *Rossby deformation radius*.

$L_R = c/f$ where $c = \sqrt{(gH)}$

What is its dependence on m?

$c = N/m$

**Extra credit**: solve in radial coordinates for a circular vortex.

**5.** Now **construct and visualize (by computer)** the balanced line vortex created by *2-vertical-mode* heating at the origin. That is, take q = Q(x=0, in distant past) $[\sin(m_1 z) - S \sin(m_2 z)]$, where $m_1 = \pi/14$km and $m_1 = 2\pi/14$km. [S may be viewed as a "stratiform fraction" when positive, or "shallow convection fraction" when negative.] Because the equation set is linear, you simply evaluate the formulas for v for the two values of m separately, and then construct the S-weighted sum of these two arrays. Also, because the equations are linear, the absolute magnitude of the total solution is arbitrary. Take $N = 2\pi/600$s (a typical tropospheric stratification).

a. **Plot the heating profiles** in the domain $z \in [0,14\text{km}]$ for S = -0.5, 0, 0.5. **Verify** that the total heat added (vertical integral) is the same in all cases.

$\sin[0,2\pi]$ has no vertical mean

b. **Calculate the Rossby radii** for the two vertical modes.

About 440 and 220 km

c. Construct numerical arrays containing vortex structures for each vertical mode in an x domain at least 6 of the larger Rossby radius in width, using your answer to part 4. **Contour the tangential wind** fields in the x-z plane for S = -0.5, 0, 0.5.

6 LR was too big – finer plots made

d. **Plot** the domain-integrated KE versus S for values of S in [-0.5,0.5]. Is the curve nonlinear? **Discuss.**

Yes the curve has a minimum at S=0 and increases with S^2. The reason is that the 2 modes have independent energy budgets. Any

excitation of the 2nd mode, of either sign, is energy that adds to the (constant in this case) energy in the 1st mode.

e. Surface flux is approximately proportional to surface wind speed. **Plot** the domain-averaged surface wind speed versus S for values of S in [-0.5,0.5]. Is the curve nonlinear? **Discuss** the implications for tropical cyclogenesis, in terms of convective heating profiles and their dependence on low-level rain evaporation and downdrafts.

The surface wind (see surface KE plot or surface wind vs. S contour plot) is weaker in the near-field environment of the vortex when the 2nd mode is active in its "stratiform" phase, with low-level cooling associated with evaporating rain and downdrafts. The surface wind is weakened in that case because the stratiform wind signature is divergence near the surface, leading to anticyclonic balanced flow. This interferes destructively with (i.e. cancels out) the convergent-cyclonic flow at low levels excited by the deep convective mode.

So, the phenomenon of (rain evaportaion / downdrafts / stratiform precipitation) causes reduced surface fluxes, which would tend to interfere with the vortex developing into a tropical cyclone through self-induced surface flux enhancements.